1

From Totally Unimodular to Balanced 0,±1 Matrices: A Family of Integer Polytopes

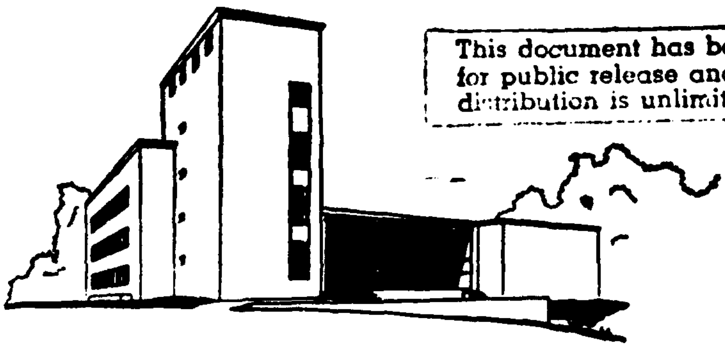
Michele Conforti[1]
Gwrard Cornuwjols[2]
Klaus Truemper[3]

Carnegie Mellon University

PITTSBURGH, PENNSYLVANIA 15213

GRADUATE SCHOOL OF INDUSTRIAL ADMINISTRATION

WILLIAM LARIMER MELLON, FOUNDER



Working Paper #1992-31

Management Science Research Report #MSRR-583

1

From Totally Unimodular to Balanced 0,$\pm$1 Matrices: A Family of Integer Polytopes

Michele Conforti[1]
Gwrard Cornuwjols[2]
Klaus Truemper[3]

July 7, 1992

This work was supported in part by NSF grants DDM-9201340, DDM-9001705 and DMS-9000376.

[1]Dipartimento di Matematica Pura ed Applicata, Università di Padova, Via Belzoni 7, 35131 Padova, Italy.

[2]Carnegie Mellon University, Schenley Park, Pittsburgh, PA 15213.

[3]University of Texas at Dallas, PO Box 830688, Richardson, TX 75083-0688.

Abstract

In this paper we introduce a family of integer polytopes and characterize them in terms of forbidden submatrices. The two extreme cases in this family arise when the constraint matrix is totally unimodular and balanced, respectively. This generalizes results of Truemper-Chandrasekaran and Conforti-Cornuéjols.

For a $0, \pm 1$ matrix A, let

$p_i(A) =$ number of $1's$ in row i,
$n_i(A) =$ number of $-1's$ in row i,
$t_i(A) =$ number of nonzeroes in row i.

Denote by $p(A)$, $n(A)$ and $t(A)$ the vectors having components $p_i(A)$, $n_i(A)$ and $t_i(A)$ respectively. We write $\mathbf{d}$ to denote a vector all of whose components are equal to d. The matrix A is *totally unimodular* if every square submatrix has determinant equal to $0, \pm 1$. The matrix A is *minimally non-totally unimodular* if it is not totally unimodular, but every proper submatrix has that property. Clearly, if a $0, \pm 1$ matrix is not totally unimodular, then it contains a minimally non-totally unimodular submatrix.

Theorem 1 (Camion [1] and Gomory (cited in [1])) *Let A be a $0, \pm 1$ minimally non-totally unimodular matrix. Then A is square, $det(A) = \pm 2$, and A^{-1} has only $\pm \frac{1}{2}$ entries. Furthermore, each row and each column of A have an even number of nonzeroes.*

Let $\mathcal{H}$ be the class of minimally non-totally unimodular matrices. Recent results of Truemper [5] (see also [6]), give a simple construction and several characterizations of all matrices in $\mathcal{H}$. Let $\mathcal{J}$ be the family of matrices that can be obtained from the identity matrix by changing some $+1's$ into $-1's$.

Theorem 2 *The following two statements are equivalent for a $0, \pm 1$ matrix A and a nonnegative integral vector c.*

(i) *A does not contain a submatrix $A' \in \mathcal{H}$ such that $t(A') \leq 2c'$, where c' is the subvector of c corresponding to the rows of A'.*

(ii) *The polytope $P(B, J, b) = \{(x, s) : \; Bx + Js = b, \; \mathbf{0} \leq x \leq \mathbf{1}, \; s \geq \mathbf{0}\}$ has integral vertices for all column submatrices B of A, all $J \in \mathcal{J}$ and all integral vectors b such that $-n(B) \leq b \leq c - n(B)$.*

Remark 3

- *When $2c \geq t(A)$, Theorem 2 gives a characterization of totally unimodular matrices which can be deduced from the Hoffman-Kruskal theorem [3].*
- *A $0, \pm 1$ matrix A is* balanced *if, in every submatrix with two nonzero entries per row and column, the sum of the entries is a multiple of four. It is easy to see that A is balanced if and only if A does not contain a submatrix $A' \in \mathcal{H}$ with $t(A') \leq \mathbf{2}$. So, when $c = \mathbf{1}$ in Theorem 2, we get a result of Conforti and Cornuéjols for $0, \pm 1$ balanced matrices (Theorem 3.3 in [2]).*
- *When A is a $0, 1$ matrix, Theorem 2 reduces to a result of Truemper and Chandrasekaran (Theorem 2 in [7]). Our proof is similar to that in [7]. Note that, here, the vertices of $P(B, J, b)$ are restricted to have $0, 1$ components x_j whereas in Theorem 2 of [7], the vertices can be general nonnegative integral vectors. But that difference is insignificant since, when A is a $0, 1$ matrix, one may duplicate columns of A to effectively eliminate the upper bound of 1 on each x_j.*

Proof of Theorem 2: (i) $\Rightarrow$ (ii). Assume the contrary and let A be a matrix of smallest order satisfying (i) but not (ii) for some nonnegative vector c. Then A has at least two rows and two columns and there exists a matrix $J \in \mathcal{J}$ such that $P(A, J, b)$ has a nonintegral vertex (x, s) for some integral vector b such that $-n(A) \leq b \leq c - n(A)$. Furthermore, for every row submatrix $\bar{A}$ of A, $P(\bar{A}, \bar{J}, \bar{b})$ is an integer polytope, where $\bar{J}$ is the corresponding submatrix of J and $\bar{b}$ is the corresponding subvector of b.

The vector x is obviously nonintegral. Furthermore $s = \mathbf{0}$, i.e. $Ax = b$, otherwise by removing a row i with $s_i > 0$, we get that $P(\bar{A}, \bar{J}, \bar{b})$ is a noninteger polytope, a contradiction. All components of x are fractional, otherwise let A^F be the column submatrix of A corresponding to the fractional components of x and A^P be the column submatrix of A corresponding to the components $x_j = 1$. Let $b^F = b - p(A^P) + n(A^P)$. Then $P(A^F, J, b^F)$ is a noninteger

polytope and b^F is an integral vector such that $-n(A^F) \leq b^F \leq c - n(A^F)$, contradicting the minimality of A. By similar reasoning, A must be square and nonsingular. Since the vector x is fractional, A cannot be totally unimodular, and hence contains a square submatrix $G = (g_{ij}) \in \mathcal{H}$.

Let i be a row of G such that $p_i(G)+n_i(G) > 2c_i$ and let $\bar{A}$ be a submatrix of A obtained by deleting a row distinct from row i. Let $\bar{b}$ be the corresponding subvector of b. The polytope $\{z : \bar{A}z = \bar{b}, \mathbf{0} \leq z \leq \mathbf{1}\}$ has precisely two vertices, say z^1 and z^2, since it is nonempty and A is nonsingular. By the minimality of A, z^1 and z^2 are $0,1$ vectors. Furthermore, $z^1 + z^2 = \mathbf{1}$ because x is a convex combination of z^1 and z^2 and all its components are fractional.

For $k = 1, 2$ define

$$L(k) = \{j : \textit{ either } g_{ij} = 1 \textit{ and } z_j^k = 1 \textit{ or } g_{ij} = -1 \textit{ and } z_j^k = 0\}.$$

Since $z^1 + z^2 = \mathbf{1}$, it follows that $|L(1)| + |L(2)| = p_i(G) + n_i(G) > 2c_i$. Assume w.l.o.g. that $|L(1)| > c_i$. Now this contradicts

$$|L(1)| = \sum_j g_{ij} z_j^1 + n_i(G) \leq b_i + n_i(A) \leq c_i,$$

where the first inequality follows from $\bar{A}z^1 = \bar{b}$.

(ii) $\Rightarrow$ (i). Let (A, c) satisfy (ii). Suppose A has m rows and contains a kxk submatrix $G \in \mathcal{H}$ such that $t(G) \leq 2c'$ where c' is the subvector of c corresponding to the rows of G. Assume w.l.o.g. that the rows and columns of G are indexed by $1, \ldots, k$ and let B be the column submatrix of A corresponding to the first k columns. Let

$$b_i = \begin{cases} \frac{p_i(G)-n_i(G)}{2} & \textit{for } i = 1, \ldots, k \\ -n_i(B) & \textit{for } i = k+1, \ldots, m. \end{cases}$$

By Theorem 1 $p_i(G) + n_i(G)$ is even for $i = 1, \ldots, k$ and therefore b is an integral vector. Furthermore $-n_i(B) \leq b_i \leq c_i - n_i(B)$. Now $P(B, -I, b)$ has a fractional vertex (x, s), where $x_i = \frac{1}{2}$ for $i = 1, \ldots, k$, $x_i = 0$ for $i = k+1, \ldots, m$, $s_i = 0$ for $i = 1, \ldots, k$, and $s_i = \frac{p_i(B)+n_i(B)}{2}$ for $i = k+1, \ldots, m$. $\square$

Consider the polytope $Q(A, c) = \{x : Ax \geq c - n(A), \mathbf{0} \leq x \leq \mathbf{1}\}$, where A is a $0, \pm 1$ matrix and c is a nonnegative integral vector. In propositional logic, problems where clause i must be satisfied at least c_i times correspond

to integer programs $max\ \{wx : x \in Q(A,c) \cap \{0,1\}^n\}$. See [4] for a survey of the connections between propositional logic and integer programming. The polytope $Q(A,c)$ has integral vertices when A and c satisfy Condition (i) of Theorem 2. Therefore the corresponding logic problems can be solved by linear programming.

References

[1] P. Camion, Characterization of totally unimodular matrices, *Proceedings of the American Mathematical Society 16* (1965) 1068-1073.

[2] M. Conforti and G. Cornuéjols, Balanced $0, \pm 1$ matrices, bicoloring and total dual integrality, preprint, Carnegie Mellon University (1992).

[3] A.J. Hoffman and J.K. Kruskal, Integral boundary points of convex polyhedra, in *Linear Inequalities and Related Systems* (H.W. Kuhn and A.W. Tucker eds.), Princeton University Press (1956).

[4] J.N. Hooker, A quantitative approach to logical inference, *Decision Support Systems 4* (1988) 45-69.

[5] K. Truemper, A decomposition theory for matroids. VII. Analysis of minimal violation matrices, to appear in *Journal of Combinatorial Theory (B)*.

[6] K. Truemper, *Matroid Decomposition,* Academic Press, Boston (1992).

[7] K. Truemper and R. Chandrasekaran, Local unimodularity of matrix-vector pairs, *Linear Algebra and its Applications 22* (1978) 65-78.

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: MSRR-583	2. GOVT ACCESSION NO
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): From Totally Unimodular to Balanced $0,\pm 1$ Matrices: A Family of Integer Polytopes	5. TYPE OF REPORT & PERIOD COVERED: Technical Report; July 7, 1992
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(S): Michele Conforti Gerard Cornuejols Klaus Truemper	8. CONTRACT OR GRANT NUMBER(S): DDM-9201340 DDM-9001705 DMS-9000376
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Graduate School of Industrial Administration Carnegie Mellon University Pittsburgh, PA 15213-3890	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS: Personnel and Training Research Programs Office of Naval Research (Code 434) Arlington, VA 22217	12. REPORT DATE: July 7, 1992
	13. NUMBER OF PAGES: 5
14. MONITORING AGENCY NAME & ADDRESS (If different from Controlling Office)	15. SECURITY CLASS (of this report)
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this paper we introduce a family of integer polytopes and characterize them in terms of forbidden submatrices. The two extreme cases in this family arise when the constraint matrix is totally unimodular and balanced, respectively. This generalizes results of Truemper-Chandrasekaran and Conforti-Cornuejols.